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Note

Spurious Solutions in Driven Cavity Calculations*

1. Introduction

When the steady Navier–Stokes equations are approximated by finite difference or finite element methods there results a coupled system of nonlinear algebraic equations. With appropriate treatment of the boundary conditions (and possibly some asymptotic approximations far from the flow region of interest) the number of algebraic equations and the number of unknowns is the same. Furthermore, since the nonlinearity in the Navier–Stokes equations is quadratic, the approximating algebraic equations are also quadratic (in any reasonable scheme). In the two-dimensional case with uniform mesh h in a domain of diameter $O(1)$ there are essentially $N^2 = 1/h^2$ unknowns and coupled quadratic equations. Now a basic result in algebraic geometry (Bezout's theorem [7, pp. 171–173]) assures us that this algebraic system has 2^{N^2} solutions, although some minor difficulties, i.e., "common intersection components," must be eliminated or else there can be manifolds of solutions. If the flow problem of interest has a unique solution, we must hope that one of these 2^{N^2} numerical solutions is a close approximation to it and that all of the others are spurious. This cursory account suggests that most of the numerical solutions are spurious!

Fortunately most of the "numerical" solutions are also complex, so real computations do not usually reveal them. Furthermore, solution procedures using continuation from known physical states may avoid them. But this is not always the case as we show in this note. Indeed even time marching schemes may lead to spurious steady states. Our results have revealed that this is particularly so when upstream differencing has been used in the driven cavity problem.

Unfortunately there is at present no good theory to determine when a solution of the approximating problem is spurious and when it is "legitimate." Indeed this imposes a severe burden on the computational fluid dynamicist to make additional tests on his results which will add weight to his assertion of their legitimacy. These tests may affirm known physical or mathematical properties of the flow or else they may assure known approximation properties of the numerical method (i.e., h^2-truncation expansion, etc.). The development of tests for legitimacy is an important and, it is hoped, a growing area which we do not study here in any detail.

We shall exhibit spurious solutions of centered difference approximations to the

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driven cavity problem. In addition we have found essentially these solutions reported in at least four papers; in only one case were they identified as being "unphysical."

Spurious solutions of Burgers' equations have been studied and computed in [10, 15]. For some channel flows they are reported in [16]. A theory assuring the absence of spurious numerical solutions for some difference methods and some equations has been developed in [3]. It is not applicable, however, to the present class of problems. Other very special studies of spurious solutions for elliptic equations with special nonlinear terms are reported in [1, 13].

2. The Driven Cavity and Approximations

The driven cavity problem seeks the steady plane incompressible flow of a viscous fluid in a square box whose lid dives the flow by moving parallel to itself with a fixed speed. In terms of a stream function ψ, such that the velocity $(u, v) = (\psi_y, -\psi_x)$, and a vorticity defined by

$$\omega \equiv u_y - v_x = \Delta^2 \psi, \tag{1a}$$

the dimensionless Navier–Stokes equations yield

$$\mathrm{Re}[\psi_y \omega_x - \psi_x \omega_y] = \Delta^2 \omega. \tag{1b}$$

Here the Reynolds number $\mathrm{Re} \equiv UL/\nu$, U is the lid speed, L is the side length and ν is the kinematic viscosity of the fluid. On eliminating ω from (1b) by using (1a) we get the scalar fourth-order equation:

$$F(\psi, \mathrm{Re}) \equiv \Delta^4 \psi - \mathrm{Re}[\psi_y \Delta^2 \psi_x - \psi_x \Delta^2 \psi_y] = 0. \tag{2}$$

The boundary conditions, $v = 0$ on four sides, $u = 0$ on three sides, and $u = 1$ on the lid, imply:

$$\begin{array}{llll} \psi = 0, & \psi_y = 1 & \text{on} & N: 0 \leqslant x \leqslant 1, \quad y = 1; \\ \psi = 0, & \psi_x = 0 & \text{on} & E: x = 1, \quad 0 \leqslant y \leqslant 1; \\ \psi = 0, & \psi_y = 0 & \text{on} & S: 0 \leqslant x \leqslant 1, \quad y = 0; \\ \psi = 0, & \psi_x = 0 & \text{on} & W: x = 0, \quad 0 \leqslant y \leqslant 1. \end{array} \tag{3}$$

To approximate the problem (1), (3) or (2), (3) we impose a square grid of mesh size $h = 1/(J-1)$ on and just exterior to the unit square Ω. Mesh functions $\{\omega_{ij}\}$ and $\{\psi_{ij}\}$ are to approximate $\{\omega(x_i, y_j)\}$ and $\{\psi(x_i, y_j)\}$, where $x_i = ih$, $y_j = jh$.

Then with the usual centered difference approximations

$$\begin{gathered} D_x^0 \psi_{ij} \equiv (\psi_{i+1,j} - \psi_{i-1,j})/2h, \qquad D_y^0 \psi_{ij} \equiv (\psi_{i,j+1} - \psi_{i,j-1})/2h \\ \Delta_h^2 \psi_{ij} \equiv (\psi_{i+1,j} - 2\psi_{ij} + \psi_{i-1,j})/h^2 + (\psi_{i,j+1} - 2\psi_{ij} + \psi_{i,j-1})/h^2 \end{gathered} \tag{4}$$

we approximate (2) by

$$F_h(\psi_{ij}, \mathrm{Re}) \equiv \Delta_h^2(\Delta_h^2 \psi_{ij}) - \mathrm{Re}[D_y^0 \psi_{ij} \Delta_h^2 D_x^0 \psi_{ij} - D_x^0 \psi_{ij} \Delta_h^2 D_y^0 \psi_{ij}] = 0. \quad (5)$$

These difference approximations are imposed at each *interior* net point $(x_i, y_j) \in \Omega$. Since fourth-order differences are employed at those net points adjacent to the boundary $\partial\Omega$ of Ω (i.e., $\partial\Omega \equiv N + E + S + W$), points on $\partial\Omega$ and points exterior to Ω but adjacent to $\partial\Omega$ also enter. The values at these points are eliminated by means of the boundary condition approximations; in place of (3) we use

$$\begin{aligned}
&\psi_{iJ} = 0, && D_y^0 \psi_{iJ} = 1, && \text{on } N_h: 1 \leqslant i \leqslant J; && j = J; \\
&\psi_{Jj} = 0, && D_x^0 \psi_{Jj} = 0, && \text{on } E_h: i = J; && 0 \leqslant j \leqslant J; \\
&\psi_{i1} = 0, && D_y^0 \psi_{i1} = 0, && \text{on } S_h: 0 \leqslant i \leqslant J, && j = 1; \\
&\psi_{1j} = 0, && D_x^0 \psi_{1j} = 0, && \text{on } W_h: i = 1, && 0 \leqslant j \leqslant J.
\end{aligned} \quad (6)$$

On eliminating the values on $\partial\Omega$ and those exterior to it by means of (6), we are left with $(J-2)^2$ equations in as many unknowns $\{\psi_{ij}\}$.

3. Solution Procedures

The reduced nonlinear difference equations of (5) are solved by means of path-following or continuation methods using Newton's method and modifications of it. The details are spelled out in [14]. Briefly, the exact Jacobian matrix $(\partial F_h/\partial\psi)$ in (5) is evaluated at the initial solution estimate $\{\psi_{ij}^{(0)}\}$ and is factored into the LU form by sparse elimination techniques (with pivoting). Then 4 or 5 iterations are performed with fixed matrix. If adequate convergence is not obtained, the Jacobian is reevaluated and factored. More than two such factorizations are never required. This iteration scheme is essentially

$$(\partial F/\partial\psi)^{(0)}\, \delta\psi^{(\nu)} = -F_h(\psi^{(\nu)}, \mathrm{Re}), \quad (7a)$$

$$\psi^{(\nu+1)}(\mathrm{Re}) = \psi^{(\nu)}(\mathrm{Re}) + \delta\psi^{(\nu)}. \quad (7b)$$

The initial guess is either

$$\psi^{(0)}(\mathrm{Re} + \delta\,\mathrm{Re}) = \overset{(\mathrm{Final})}{\psi\,(\mathrm{Re})}, \quad (8a)$$

or else

$$\psi^{(0)}(\mathrm{Re} + \delta\,\mathrm{Re}) = \overset{(\mathrm{Final})}{\psi\,(\mathrm{Re})} + \delta\,\mathrm{Re}(\partial\overset{(\mathrm{Final})}{\psi\,(\mathrm{Re})}/\partial\,\mathrm{Re}). \quad (8b)$$

The latter is more accurate and allows larger steps in $\delta\,\mathrm{Re}$. Even more accurate initial estimates (using Hermite extrapolation) have been used, see [14].

TABLE I

J	LU-factorization	Back-solve
20	0.23	0.028
30	0.83	0.069
40	1.90	0.130

These continuation procedures are easily modified to compute bifurcations or limit point behavior by introducing arclength continuation procedures, see Keller [9]. In brief, we compute both $\{\psi_{ij}(s)\}$ and $\mathrm{Re}(s)$ in terms of some arclength-like parameter s. Then if $\psi(\mathrm{Re})$ becomes double valued, i.e., the solution path and hence $\mathrm{Re}(s)$ turns back on itself, the computations in terms of s have no difficulty. But in terms of increasing Re, the problem may have no solution. The extra work in arclength continuation is trivial (one extra back-solve for each iteration), see [9, 14].

The computations were done on the CDC STAR-100 (i.e., CYBER 203) at Arden Hills, Minn. Sample CPU times in seconds, for three indicated meshes are given in Table I. The entire path of solutions on the 40×40 grid (see Fig. 1), involving 79 different values of Re in $800 \leqslant \mathrm{Re} \leqslant 7686$, required 325 seconds of CPU time.

The solution branches for $J = 30$ and $J = 40$ were traced to higher Reynolds number and no other limit points were found. For $J = 20$, however, a limit point at high Reynolds number does occur (see Fig. 4). The wide difference between the course and fine grid results persists. At $\mathrm{Re} = 7686$ we found $\psi = -0.0182$ at the vortex center on the 40×40 grid. By comparison, Ghia, Ghia, and Shin, using a 256×256 grid, found the value -0.120 [6].

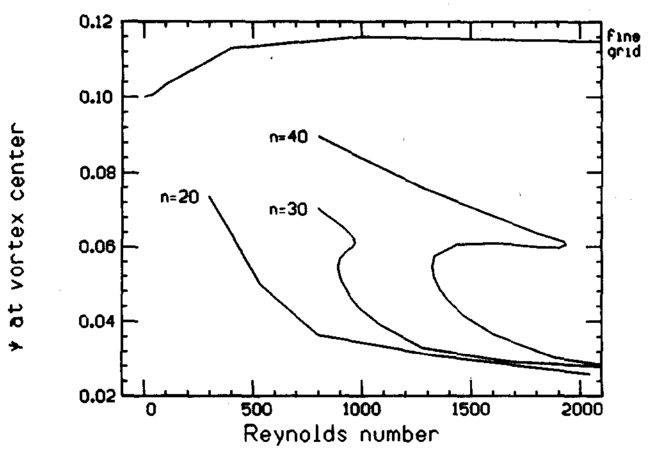

FIG. 1. Difference scheme solution branches.

4. Results and Spurious Solutions

We originally sought to investigate the occurrence of bifurcation and/or limit point behavior in the driven cavity problem. Typically in such calculations the numerical methods have failed to converge at some Re value for which the solution seems quite reasonable. Thus we suspected limit points. To find them, we planned to use very coarse grids, as the computations are expensive, and then to refine the grids near the limit points to get accurate results. We did indeed find limit points on coarse grids and in fact three nonunique solutions. But on refining the grids, the limit points seem to move off to infinity. The multiple solutions thus disappear and we retain only one solution. If this described behavior is correct, then we have indeed found spurious solutions to the difference equations.

In more detail, we computed paths of solutions $[\{\psi_{ij}(s)\}, \text{Re}(s)]$ on nets with $J = 20$, 30, 40, and 50. No limit points were observed for $J = 20$. In the other cases the solution paths seemed to form S-shaped curves with two limit points. These can be seen in Fig. 1 where we plot $\max_{i,j} |\psi_{ij}(\text{Re})|$ against Re for several grids. The values of the Reynolds numbers at which the limit points are located are given in Table II. We did not bother to get the second limit point for $J = 50$. Upon refining the net to $J \geqslant 100$ (see [14]) all the limit points disappeared. These fine grid calculations agree very well with other recent results [2, 17]. Table II clearly suggests that $\text{Re}_1(J) \to \infty$ as $J \to \infty$. Indeed the growth is superlinear.

In Figs. 2 and 3 we show streamline plots of the spurious solutions on the $J = 40$ grid. Figure 2 is a spurious solution between the limit points and Fig. 3 is a solution on the spurious branch beyond the limit point at Re_2. Solutions qualitatively very similar to these have been reported in the literature [4–6, 12]. Several authors have observed that these solutions are "nonphysical" but there has been no previous indication of their origin.

Olsen and Tuann [11] used a finite element approximation of (2) with conforming reduced quintic elements having 18 degrees of freedom on a triangular grid with $h = \frac{1}{8}$. With Newton's method and continuation in Re they could not obtain solutions beyond $\text{Re} > 3450$. This is reasonably close to our limit point at $\text{Re}_1 = 3981$ for $J = 50$. Our $O(h^2)$ accuracy ($h = 1/49$) is somewhat better than their accuracy with quintic elements for this case.

We also used the centered difference form of (1a), (1b) but with higher order

TABLE II

J	Re_1	Re_2
30	968.5	892.6
40	1936.0	1322.0
50	3981.0	—

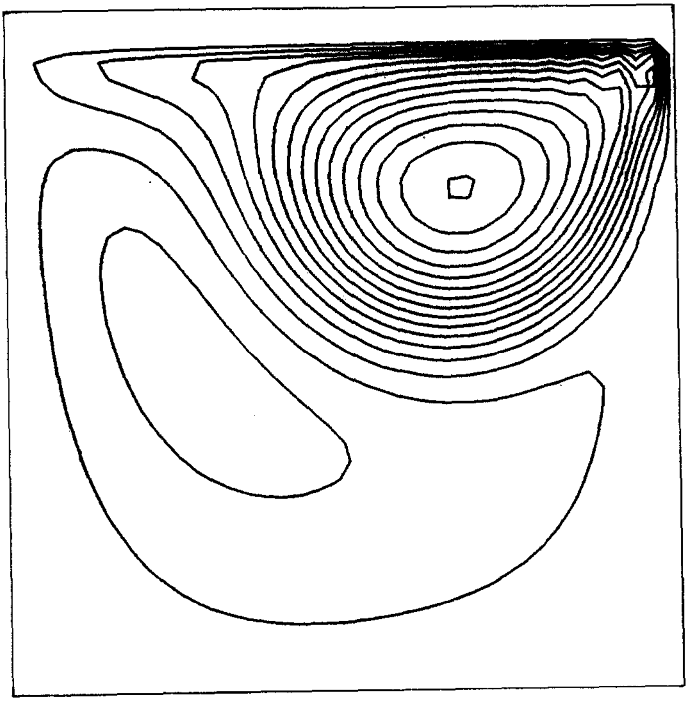

FIG. 2. Streamlines at Re = 1325, $J = 40$.

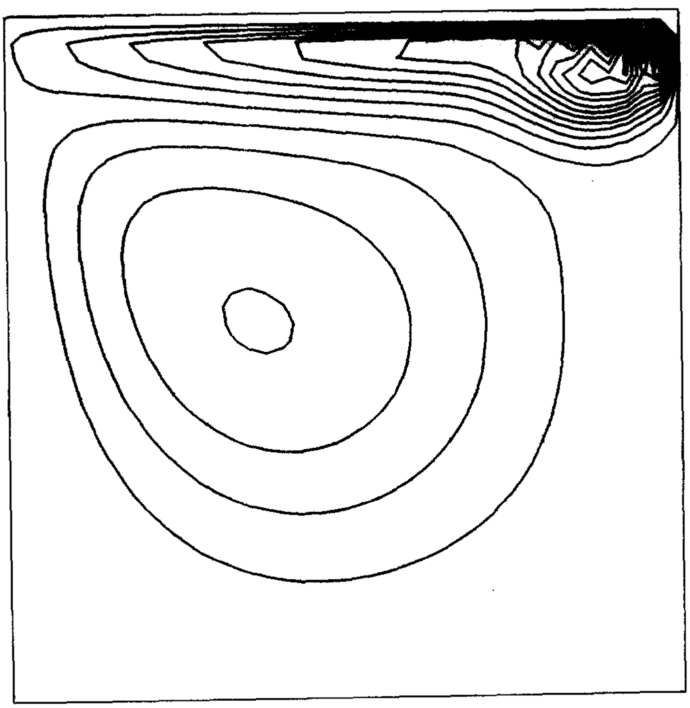

FIG. 3. Streamlines at Re = 2822, $J = 40$.

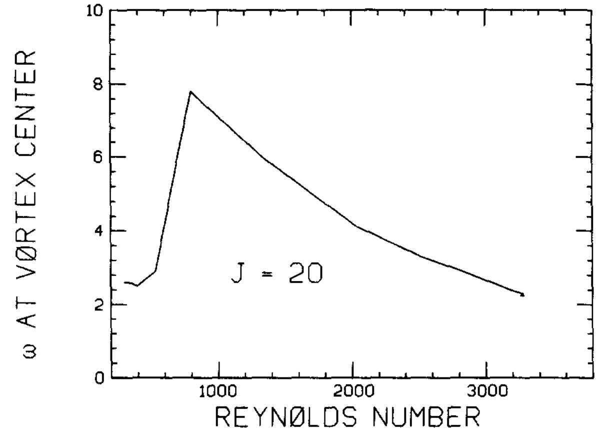

FIG. 4. Vorticity versus Re for $J = 20$.

approximations of the normal derivatives in the boundary condition. Thus, for example, on side W we employed:

$$\frac{\partial \psi}{\partial x}(0, y) \approx \frac{1}{6h}\,[-2\psi(-h, y) + 6\psi(h, y) - \psi(2h, y)],$$

which has $O(h^3)$ truncation error. The resulting discrete equations were solved on the $J = 30$ grid and a limit point was found at $\mathrm{Re}_1 = 962.5$. Compared to the value $\mathrm{Re}_1 = 968.5$ from our centered equations, we conclude that the spurious solutions are unfortunately persistent and do not change radically with "small" changes in the difference formulation!

Figures 4 and 5 show the vorticity at the vortex center as a function of Re for

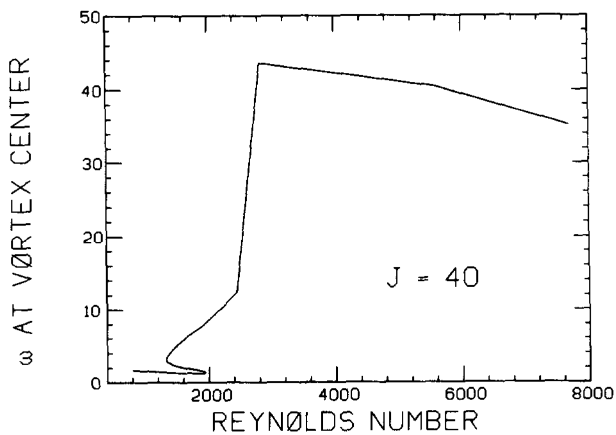

FIG. 5. Vorticity versus Re for $J = 40$.

$J = 20$ and 40. In both cases the vorticity takes a sudden jump to several times its correct value, followed by a slow decay. This is probably due to the collapse of the main vortex evident in Fig. 3. It might be a useful indicator of loss of accuracy.

It would perhaps be interesting to compute *all* solutions of the difference equations as Shubin, Stephens, and Glaz [15] have done for a discretization of Burgers' equation. Unfortunately, the 2^{N^2} growth in the number of solutions as the mesh size is reduced does not allow much hope for this to be done in any realistic case.

REFERENCES

1. E. ALLGOWER, *in* "Proc. Conf. Roy. Irish Acad.," New York/London, 1975.
2. A S. BENJAMIN AND V. E. DENNY, *J. Comput. Phys.*, **13** (1979), 340–358.
3. W.-J. BEYN AND E. J. DOEDEL, *SIAM J. Sci. Statist. Comput.* **2** (1981), 107–120.
4. J. D. BOZEMAN AND C. DALTON, *J. Comput. Phys.* **12** (1973), 348–363.
5. T. J. CHUNG, "Finite Element Analyses in Fluid Dynamics," McGraw–Hill, New York, 1978.
6. U. GHIA, K. N. GHIA AND C. T. SHIN, *in* "Symposium on Multigrid Methods," NASA-Ames Research Center, Moffett Field, Calif. 1981.
7. P. GRIFFITHS AND J. HARRIS, "Principles of Algebraic Geometry," Wiley–Interscience, New York, 1978.
8. M. M. GUPTA AND R. P. MANOHAR, *Int. J. Numer. Methods Engrg.* **15** (4) (1980), 557–573.
9. H. B. KELLER, *in* "Applications of Bifurcation Theory (P. H. Rabinowitz, Ed.), Academic Press, New York, 1977.
10. R. B. KELLOGG, G. R. SHUBIN, AND A. B. STEPHENS, *SIAM J. Numer. Anal.* **17** (6) (1980), 733–739.
11. M. D. OLSON AND S.-Y. TUANN, *Comput. and Fluids* **7** (1979), 123–135.
12. M. NALLASAMY AND K. K. PRASAD, *J. Fluid Mech.* **72** (2) (1977), 391–414.
13. H.-O. PEITGEN AND K. SCHMITT, "Positive and spurious solutions of nonlinear eigenvalue problems," Report NR. 42, Univ. Bremen, May 1981.
14. R. SCHREIBER AND H. B. KELLER, *J. Comput. Phys.*, submitted.
15. A. B. STEPHENS AND G. R. SHUBIN, *SIAM J. Sci. Statist Comput.* **2** (1981), 404–415.
16. G. R. SHUBIN, A. B. STEPHENS, AND H. M. GLAZ, *J. Comput. Phys.*, to appear.
17. K. H. WINTERS AND K. A. CLIFFE, A Finite Element Study of Driven Laminar Flow in a Square Cavity, UKAERE Harwell Report R. 9444, Harwell, England, 1979.

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